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Rational Approximation to x^α

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In [1], Newman showed the existence of rational functions $R_n(x)$ of order n (see below) for which

$$\max_{0 \leqslant x \leqslant 1} | x^{1/2} - R_n(x)| = O(\exp(-cn^{1/2})), \tag{1}$$

with an absolute positive constant c, and he proved also that there does not exist any such rational $r_n(x)$ for which

$$\max_{0 \leqslant x \leqslant 1} | x^{1/2} - r_n(x)| \leqslant (1/2) \exp(-cn^{1/2}). \tag{2}$$

He stated that an analog of (1) holds for x^α ($\alpha > 0$ rational), i.e., for some such rational function $R_n(x, \alpha)$,

$$\max_{0 \leqslant x \leqslant 1} | x^\alpha - R_n(x, \alpha)| = O(\exp(-c(\alpha)\, n^{1/2})), \ n \geqslant n_0(\alpha)$$

where $c(\alpha) > 0$ and $n_0(\alpha)$ depends only on α.

In 1967 [4], Freud and Szabados obtained a weaker result with $n^{1/3}$ instead of $n^{1/2}$. Goncar [7] proved Newman's statement in 1967.

In this paper, we give another proof of Newman's statement and also prove that this is the best possible result.

We use almost the same notation as Newman [1].

NOTATION. n and s are positive integers, $\zeta = \exp(-n^{-1/2})$,

$$p(x) = \prod_{k=0}^{n-1} (x + \zeta^k), \qquad q(x) = xp(x)$$

and

$$r(x) = \sum_{k=0}^{s-1} q(\epsilon^k x) \Big/ \sum_{k=0}^{s-1} p(\epsilon^k x),$$

where $\epsilon = \exp(2\pi i/s)$.

The order of a rational function is defined as the maximum of the degrees of its numerator and denominator.

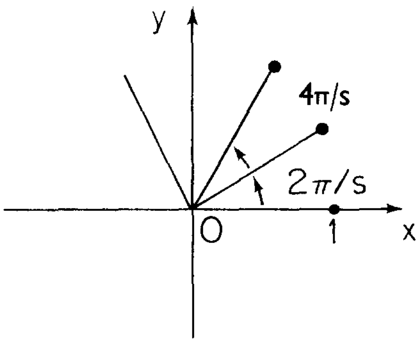
THEOREM. (I) $|x^{1/s} - r(x)| \leqslant A \exp(-Bn^{1/2})$ *throughout* $[0, 1]$, *where* A *and* B *depend only on* s.

(II) *There exists a constant* C *with the property that there does not exist an* nth-*order rational function* $R(x)$ *such that*

$$|x^{1/s} - R(x)| \leqslant (1/2) \exp(-Cn^{1/2}) \text{ throughout } [0, 1]. \tag{4}$$

The proof will use the technique of Newman [1] with some necessary modifications.

DEFINITION. By the s-star, we mean the union of s closed unit segments with one common endpoint the origin, and equiangled. One of the segments must be $[0, 1]$.



For example, the 2-star is the interval $[-1, 1]$.

Remark. The approximation of $x^{1/s}$ in $[0, 1]$ is equivalent to the approximation of $|x|$ in the s-star. In fact, if $R(x)$ approximates $x^{1/s}$ in $[0, 1]$, then $R(x^s)$ approximates $(x^s)^{1/s}$ in the s-star; and conversely: If $R(x)$ approximates $|x|$ in the s-star, then

$$\frac{1}{s} \sum_{k=0}^{s-1} R(\epsilon^k (x)^{1/s}) \tag{5}$$

approximates $x^{1/s}$ on $[0, 1]$. Thus, the orders of approximation of $x^{1/s}$ and of $|x|$ are the same.

LEMMA 1. *For* $c \exp(-n^{1/2}) \leqslant x \leqslant 1, 0 < c < 1,$ *and* $1 \leqslant l \leqslant s - 1$, *there exist a constant* c_1 *such that*

$$\left| \frac{p(\epsilon^l x)}{p(x)} \right| \leqslant \exp(-c_1 n^{1/2}). \tag{6}$$

Proof.

$$\left|\frac{p(\epsilon^l x)}{p(x)}\right|^2 = \prod_{k=0}^{n-1}\left|\frac{\epsilon^l x + \zeta^k}{x + \zeta^k}\right|^2 = \prod_{k=0}^{n-1}\frac{x^2 + \zeta^{2k} - Kx\zeta^k}{x^2 + \zeta^{2k} + 2x\zeta^k}.$$

Here, $K = 2\cos(2\pi l/s)$, and hence $|K| \leqslant 2$. Now,

$$\frac{x^2 + \zeta^{2k} - Kx\zeta^k}{x^2 + \zeta^{2k} + 2x\zeta^k} = 1 - \frac{(2 + K)\, x\zeta^k}{(x + \zeta^k)^2}.$$

This function has only one local minimum at $x = \zeta^k$. Let us take

$$\zeta^{j+1} \leqslant x \leqslant \zeta^j,\ 0 \leqslant j \leqslant n.$$

Then,

$$\left|\frac{p(\epsilon^l x)}{p(x)}\right|^2 \leqslant \prod_{k=0}^{j}\frac{\zeta^{2n} + \zeta^{2k} - K\zeta^{n+k}}{\zeta^{2n} + \zeta^{2k} + 2\zeta^{n+k}}\prod_{k=j+1}^{n-1}\frac{\zeta^{2j} + \zeta^{2k} - K\zeta^{j+k}}{\zeta^{2j} + \zeta^{2k} + 2\zeta^{j+k}}$$

$$= \prod_{m=1}^{n}\frac{\zeta^{2m} + 1 - K\zeta^m}{(\zeta^m + 1)^2} = \prod_{m=1}^{n}\left[1 - \frac{(2 + K)\,\zeta^m}{(\zeta^m + 1)^2}\right]$$

$$\leqslant \exp\left[-(2 + K)\sum_{m=1}^{n}\frac{\zeta^m}{(\zeta^m + 1)^2}\right] \leqslant \exp\left[-\frac{(2 + K)}{4}\sum_{m=1}^{n}\zeta^m\right]$$

$$= \exp\left[-\frac{(2 + K)}{4}\frac{1 - \zeta^n}{1 - \zeta}\right].$$

In [1], Newman pointed that for $n > 4$, $2\zeta(1 - \zeta^n) > 1$, and also

$$1/(1 - \zeta) \geqslant n^{1/2}.$$

Using these facts, we obtain the result.

LEMMA 2. *Let $b \geqslant a \geqslant 0$, let ξ be any complex number, and let $s > k \geqslant 0$ be two integers such that $\sin(2\pi k/s) \geqslant 0$. Then, there exists a positive constant D that depends only on s and k such that*

$$\int_a^b \log\left|\frac{\epsilon^k t + \xi}{t - \xi}\right|\frac{dt}{t} \geqslant -D. \tag{7}$$

Proof. The proof is similar to that of Newman [1, Lemma 3]. Denote $\xi = u + iv$ and $\epsilon^k = u' + iv'$. Then, for $t \geqslant 0$,

$$\left|\frac{\epsilon^k t + \xi}{t - \xi}\right| = \left(\frac{(tu' + u)^2 + (tv' + v)^2}{(t - u)^2 + v^2}\right)^{1/2} \geqslant \left|\frac{|tu'| - |u|}{t + |u|}\right|. \tag{8}$$

If $u = 0$, there is nothing to prove. We have

$$\int_a^b \log \left| \frac{\epsilon^k t + \xi}{t - \xi} \right| \frac{dt}{t}$$

$$\geqslant \int_a^b \log \left| \frac{t \mid u' \mid - \mid u \mid}{t + \mid u \mid} \right| \frac{dt}{t} = \int_{a/|r|}^{b/|u|} \log \left| \frac{t \mid u' \mid - 1}{t + 1} \right| \frac{dt}{t}$$

$$\geqslant \int_0^\infty \log \left| \frac{t \mid u' \mid - 1}{t + 1} \right| \frac{dt}{t}. \tag{9}$$

Here, $u' = \sin(2k\pi/s)$. The last expression is a negative constant that depends only on s and k.

LEMMA 3. *Let $\tilde{p}(x) \not\equiv 0$ be any nth degree complex polynomial. There exists a point x in* $[\exp(-n^{1/2}), 1]$, *where*

$$\left| x \frac{\tilde{p}(\epsilon^k x)}{\tilde{p}(x)} \right| > \exp[-1 - Dn^{1/2}]. \tag{10}$$

(Here, we assume $\operatorname{Im}(\epsilon^k) \geqslant 0$, $\operatorname{Re}(\epsilon^k) \leqslant 0$ and D is the constant that appears in Lemma 2).

This Lemma generalizes Newman [1, Lemma 4].

Proof. Let $\delta = \exp(-n^{1/2})$. Then,

$$\int_\delta^1 \log \left| t \frac{\tilde{p}(\epsilon^k t)}{\tilde{p}(t)} \right| \frac{dt}{t} = \int_\delta^1 \frac{\log t}{t} dt + \sum_\xi \int_\delta^1 \log \left| \frac{\epsilon^k t + \xi}{t - \xi} \right| \frac{dt}{t}, \tag{11}$$

where ξ takes the values of the zeros of $\tilde{p}(t)$.

Since $\int_\delta^1 (\log t/t)\, dt = -n/2$, using Lemma 2 we obtain

$$\int_\delta^1 \log \left| t \frac{\tilde{p}(\epsilon^k t)}{\tilde{p}(t)} \right| \frac{dt}{t} \geqslant -\frac{n}{2} - nD. \tag{12}$$

If the lemma were false, we would have

$$\int_\delta^1 \log \left| t \frac{\tilde{p}(\epsilon^k t)}{\tilde{p}(t)} \right| \frac{dt}{t} \leqslant (-1 - D)\, n^{1/2} \int_\delta^1 \frac{dt}{t} = (-1 - D)\, n, \tag{13}$$

a contradiction. Q.E.D.

Proof of Part (I) of the Theorem. We shall prove the theorem for $|x|$ instead of for $x^{1/s}$. It is clear that it is enough to prove the theorem only for $0 \leqslant x \leqslant 1$.

For $0 \leqslant x \leqslant c \exp(-n^{1/2})$ we need only to evaluate $|r(x)|$. From the definition of $r(x)$, if $x \geqslant 0$, also, $r(x) \geqslant 0$. If we write $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, we obtain

$$r(x) = \frac{a_{s-1}x^s + a_{2s-1}x^{2s} + \cdots}{a_0 + a_s x^s + a_{2s}x^{2s} + \cdots}. \tag{14}$$

The denominator has at least the same number of terms as the numerator. Also

$$\begin{aligned} a_n &= 1 \\ a_l &= \sum_{0 \leqslant k_0 < k_1 < \cdots < k_{n-l-1} \leqslant n-1} \zeta^{k_0 + k_1 + \cdots + k_{n-l-1}}, \qquad (l = 0, 1, \ldots, n-1), \end{aligned} \tag{15}$$

$$r(x) = x\,\frac{a_{s-1}x^{s-1} + a_{2s-1}x^{2s-1} + \cdots}{a_0 + a_s x^s + \cdots},$$

and

$$r(x) \leqslant x \max_{j \geqslant 1} \frac{a_{js-1}x^{js-1}}{a_{(j-1)s}x^{(j-1)s}}. \tag{16}$$

The a's are symmetric polynomials. For every term in the denominator of the last fraction, we obtain in the numerator at least $\binom{n}{s-1}$ terms by omitting $s-1$ factors. Multiplying by x^{s-1}, with $x \leqslant c \exp(-n^{1/2})$, we find that the ratio is bounded by $\binom{n}{s-1}$, and we obtain

$$r(x) \leqslant \binom{n}{s-1} c^{-s} \exp(-sn^{1/2}). \tag{17}$$

Choosing different constants, we obtain

$$r(x) \leqslant A \exp(-Bn^{1/2}), \tag{18}$$

with A and B independent of n and $0 < B < 1$. Now, let $c \exp(-n^{1/2}) \leqslant x \leqslant 1$. Then, using Lemma 1,

$$\begin{aligned} ||x| - r(x)| &= x \left| \frac{\sum_{k=1}^{s-1} p(\epsilon^k x) - \sum_{k=1}^{s-1} \epsilon^k p(\epsilon^k x)}{\sum_{k=0}^{s-1} p(\epsilon^k x)} \right| \\ &\leqslant 2x \frac{\sum_{k=1}^{s-1} |p(\epsilon^k x)|}{|\sum_{k=0}^{s-1} p(\epsilon^k x)|} \leqslant 2x \frac{\sum_{k=1}^{s-1} |p(\epsilon^k x)/p(x)|}{1 - \sum_{k=1}^{s-1} |p(\epsilon^k x)/p(x)|} \\ &\leqslant 2 \frac{\sum_{k=1}^{s-1} \exp(-c_1 n^{1/2})}{1 - \sum_{k=1}^{s-1} \exp(-c_1 n^{1/2})} = 2 \frac{(s-1)\exp(-c_1 n^{1/2})}{1 - (s-1)\exp(-c_1 n^{1/2})}, \end{aligned}$$

Thus, for $n \geqslant n_0(s)$,

$$||x| - r(x)| \leqslant A \exp(-Bn^{1/2})$$

for appropriate positive constants A, B.

Proof of Part II of the Theorem. Let $C = (1 + D)s^{1/2}$, where D is the constant in Lemma 2. Assume that there exists an $R(x)$ satisfying

$$||x| - R(x)| \leqslant \exp(-Cn^{1/2}). \tag{19}$$

Set

$$R_1(x) = \frac{\sum_{k=0}^{s-1} R(\epsilon^k x)}{s} - R(0), \tag{20}$$

a rational function of order sn with $R_1(0) = 0$. In fact,

$$R_1(x) = x^s \frac{S(x^s)}{Q(x^s)}, \qquad Q \text{ and } S \text{ polynomials.} \tag{21}$$

From (19), we have for x in the s star,

$$R_1(x) \geqslant |x| - \exp(-Cn^{1/2}). \tag{22}$$

For $x > \exp(-Cn^{1/2})$ we have $R_1(x) > 0$ and $S(x^s)\, Q(x^s) > 0$; hence, we can assume that $S(x^s) > 0$ and that $Q(x^s) > 0$ for $x > \exp(-Cn^{1/2})$.

Now,

$$|x - R_1(x)| = \left| x - x^s \frac{S(x^s)}{Q(x^s)} \right| = \left| x \frac{Q(x^s) - x^{s-1}S(x^s)}{Q(x^s)} \right|$$
$$\geqslant \left| x \frac{Q(x^s) - x^{s-1}S(x^s)}{Q(x^s) - \epsilon^k x^{s-1}S(x^s)} \right|$$

and $\operatorname{Re}(\epsilon^k) \leqslant 0$, $\operatorname{Im}(\epsilon^k) > 0$. We apply Lemma 3 to the polynomial

$$Q(x^s) - \epsilon^k x^{s-1}S(x^s) \tag{23}$$

and obtain that for some x, $\exp(-(sn)^{1/2}) \leqslant x \leqslant 1$,

$$\left| x \frac{Q(x^s) - x^{s-1}S(x^s)}{Q(x^s) - x^{s-1}\epsilon^k S(x^s)} \right| > \exp(-Cn^{1/2}),$$

a contradiction. Q.E.D.

It would be interesting to obtain the best values of the constants A, B, and C.

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